

THE DISTRIBUTION OF PERMUTATION MATRIX ENTRIES UNDER RANDOMIZED BASIS

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ABSTRACT. We study the distribution of entries of a random permutation matrix under a “randomized basis,” i.e. we conjugate the random permutation matrix by an independent random orthogonal matrix drawn from Haar measure. It is shown that under certain conditions, the linear combination of entries of a random permutation matrix under a “randomized basis” converges to a sum of independent variables $sY + Z$ where Y is Poisson distributed, Z is normally distributed, and s is a constant.

1. INTRODUCTION

Traditionally, random matrix theory has focused primarily on the eigenvalue distributions of matrices drawn from various measures (see e.g. [8] for a quick survey or [1] for an introduction to the field). However, there has also been interest in studying statistics related to entries of random matrices. For example, the study of the entries of a Haar-distributed orthogonal matrix is a classical subject originated by Borel [2] who showed that the normalized first coordinate of a random vector chosen from the n -dimensional sphere converges in distribution to a standard normal. Diaconis and Freedman [10] strengthened this to obtain total variation convergence of the first k coordinates to independent standard normals when $k = o(n)$. Jiang [16] then further strengthened this result considerably to show that if $p_n = o(\sqrt{n})$ and $q_n = o(\sqrt{n})$, then the total variation distance between the joint distribution of the entries of the upper left $p_n \times q_n$ block of a Haar-distributed orthogonal $n \times n$ matrix and $p_n q_n$ independent standard normals converges to 0. Moreover, he proved this result is sharp.

Matrix entries of random permutation matrices have also been studied. Hoeffding’s combinatorial central limit theorem [15] gives a result on linear combinations of random permutation matrix entries. Let A_n be a sequence of $n \times n$ real matrices such that $\text{Tr}(A_n A_n^T) = n$. Hoeffding shows that under certain conditions on A_n , if P is a random $n \times n$ permutation matrix, then $\text{Tr}(A_n P)$ converges weakly to a standard normal random variable as $n \rightarrow \infty$.

In this article, we study the distribution of entries of a random permutation matrix in a basis free way. Individual entries of a permutation matrix are either 0 or 1, but if we conjugate by a random orthogonal matrix, we can look at the distribution of entries under a “randomized basis.” To formalize this, let P be an $n \times n$ permutation matrix drawn uniformly from the symmetric group \mathfrak{S}_n and let M be a Haar-distributed $n \times n$ orthogonal matrix independent of P . (With little confusion, we will often conflate a permutation matrix and its corresponding permutation in \mathfrak{S}_n .) Let A_n be a sequence of $n \times n$ real matrices such that $\text{Tr}(A_n A_n^T) = n$. Then

it is shown in Theorem 3.2 below that under certain conditions on A_n ,

$$\text{Tr}(A_n M P M^T) \xrightarrow{d} sY + Z$$

as $n \rightarrow \infty$ where Y and Z are independent random variables such that Y is Poisson distributed, Z is normally distributed, and s is a constant. Whereas Hoeffding's combinatorial CLT shows convergence for linear combinations of random permutation matrix entries, Theorem 3.2 can be interpreted as a distributional convergence for linear combinations of random permutation matrix entries under a "randomized basis."

We briefly review some other related results. Let A be an $n \times n$ real (nonrandom) matrix such that $\text{Tr}(AA^T) = n$. D'Aristotile, Diaconis, and Newman [6] have shown using characteristic function methods that if M is an $n \times n$ random orthogonal matrix, then $\text{Tr}(AM)$ converges in distribution to a standard normal random variable as $n \rightarrow \infty$ uniformly in A . Recently, Meckes [19] has improved this result. She shows that the total variation distance between $\text{Tr}(AM)$ and a standard normal random variable is bounded by $\frac{2\sqrt{3}}{n-1}$ and that this rate is sharp up to a constant. In [3], Chatterjee and Meckes obtain bounds on the Wasserstein distance between the multivariate distributions $(\text{Tr}(A_1 M), \text{Tr}(A_2 M), \dots, \text{Tr}(A_k M))$ and a Gaussian random vector. Their techniques involve generalizing Stein's method of exchangeable pairs.

2. LIMITING DISTRIBUTION OF A SINGLE ENTRY

In this section, we start by showing that a single scaled matrix entry of MPM^T converges in distribution to a standard normal distribution. Later sections will consider joint distributions of the matrix entries.

Note that the distribution of MPM^T where P is random can be thought of as a mixture of distributions of MPM^T where P is a fixed permutation matrix. Thus, we first assume P is fixed. If P is similar to another permutation matrix Q via an orthogonal transformation V , i.e. $P = VQV^T$, then $MPM^T = (MV)Q(MV)^T \stackrel{d}{=} MQM^T$. Thus, the distribution of MPM^T only depends on the cycle type of the permutation matrix P .

First, we consider the case where the permutation consists of the single n -cycle $(1, 2, \dots, n)$. Let C_n be the corresponding permutation matrix. Then

$$(MC_n M^T)_{ab} = \sum_{i=1}^n M_{a,i} M_{b,i+1}$$

(where abusing notation slightly, $M_{b,n+1} := M_{b,1}$) By symmetry, it is clear that $\mathbb{E}[M_{ij}^2] = 1/n$. Moreover, $\sqrt{n}(M_{a,i}, M_{b,i+1}) \xrightarrow{d} (X_1, X_2)$ where X_1 and X_2 are i.i.d $\mathcal{N}(0, 1)$. (The convergence is actually in total variation, see e.g. [10, 16]). Therefore,

$\sum_{i=1}^n M_{a,i} M_{b,i+1}$ should have variance on the order of $1/n$ and it is natural to consider the scaled random variable $\sqrt{n}(MC_n M^T)_{ab}$.

To obtain the limiting distribution of $\sqrt{n}(MC_n M^T)_{ab}$ (as well as the limiting joint distribution of multiple entries derived later on), the key idea will be to use a martingale central limit theorem for dependent random variables proved by McLeish [18]. Specifically, we use his Corollary (2.13). (See [14] for a book-length treatment on martingale limit theorems.) Let $\{X_{n,i}; 1 \leq i \leq n\}$ be a triangular array of

random variables. Define $\sigma_{n,i}^2 = \mathbb{E}[X_{n,i}^2]$, $s_n^2 = \sum_{i=1}^n \sigma_{n,i}^2$, and $S_n = \sum_{i=1}^n X_{n,i}$. Let $\mathcal{F}_{n,i} = \sigma(X_{n,1}, X_{n,2}, \dots, X_{n,i})$. We say that $X_{n,i}$ is a martingale difference array if $\mathbb{E}(X_{n,i} | \mathcal{F}_{n,i-1}) = 0$. Then we have the following:

Theorem 2.1 (McLeish). *Suppose $X_{n,i}$ is a martingale difference array normalized by its variance s_n^2 , satisfying the Lindeberg condition $\lim_{n \rightarrow \infty} \sum_{i=1}^n \int_{|X_{n,i}| > \varepsilon} X_{n,i}^2 dP = 0$ for all $\varepsilon > 0$ and the condition $\limsup_{n \rightarrow \infty} \sum_{i \neq j} \mathbb{E}[X_{n,i}^2 X_{n,j}^2] \leq 1$. Then $S_n \xrightarrow{d} \mathcal{N}(0, 1)$.*

The following lemma will also be useful in the sequel:

Lemma 2.1. *Let the random variable $X_n = n^k M_1^{k_1} \dots M_r^{k_r}$ where $k_1 + \dots + k_r = 2k$ and M_i are distinct entries of the $n \times n$ Haar distributed orthogonal matrix. Define $Y = Z_1^{k_1} \dots Z_r^{k_r}$ where Z_i are i.i.d. standard normals. Then $X_n \xrightarrow{L^1} Y$.*

Proof. By [16], the joint distribution of any k fixed entries (scaled by \sqrt{n}) of a Haar distributed orthogonal matrix converges in total variation to the joint distribution of k i.i.d. standard normals. In particular, by the continuous mapping theorem, we see that $X_n \xrightarrow{d} Y$. To get L^1 convergence, we need uniform integrability, which is implied by $\limsup_{n \rightarrow \infty} \mathbb{E}[X_n^2] < \infty$. By the Cauchy-Schwarz inequality, it is sufficient to show that $\limsup_{n \rightarrow \infty} \mathbb{E}[n^{2k} M_1^{4k}] < \infty$. But M_1^2 has a Beta distribution with parameters $(1/2, (n-1)/2)$ (see e.g. [20, pp. 145]) and hence $\mathbb{E}[n^{2k} M_1^{4k}] = O(1)$. \square

Remark 2.1. Lemma 2.1 gives rather crude asymptotics for the expectation of products of Haar distributed matrix entries. Collins and Sniady [5] have developed machinery using Weingarten functions to compute the integrals of polynomial functions of Haar distributed orthogonal, unitary, and symplectic entries for any dimension n . (Please refer to [5] for the definition of Weingarten functions. We will only require the asymptotics of Weingarten functions mentioned below.)

For the orthogonal group, Collins and Sniady's formula says that the integral over Haar measure $\int_{M \in O(n)} M_{i_1 j_1} \dots M_{i_{2r} j_{2r}} dM$ is the sum of Weingarten functions $Wg(\mathbf{m}, \mathbf{n})$ over pair partitions \mathbf{m} and \mathbf{n} of the set $\{1, \dots, 2r\}$ such that $i_{\mathbf{m}(2k-1)} = i_{\mathbf{m}(2k)}$ and $j_{\mathbf{n}(2k-1)} = j_{\mathbf{n}(2k)}$ where $\mathbf{m} = \{\{\mathbf{m}(1), \mathbf{m}(2)\}, \dots, \{\mathbf{m}(2r-1), \mathbf{m}(2r)\}\}$ and $\mathbf{n} = \{\{\mathbf{n}(1), \mathbf{n}(2)\}, \dots, \{\mathbf{n}(2r-1), \mathbf{n}(2r)\}\}$. In other words, \mathbf{m} represents pairing up matrix entries in the same row and \mathbf{n} represents pairing up entries in the same column. In particular, the expectation is non-zero only if there are an even number of entries in each row and column.

In [4], Collins and Matsumoto obtain a simpler formula for the orthogonal Weingarten function that significantly reduces the complexity involved in the computation of Weingarten formulas and show that the integrals are always rational functions of the dimension n . They also provide asymptotics for the Weingarten function:

$$Wg(\mathbf{m}, \mathbf{n}) = O(1/n^{-2r+l(\mathbf{m}, \mathbf{n})})$$

Here, the length $l(\mathbf{m}, \mathbf{n})$ is defined as follows: consider the graph with vertex set $1, 2, \dots, 2r$ and edge set that consists of $\{\mathbf{m}(2k-1), \mathbf{m}(2k)\}, \{\mathbf{n}(2k-1), \mathbf{n}(2k)\}$ where

$1 \leq k \leq r$. Then $l(\mathbf{m}, \mathbf{n})$ is the number of components in this graph. Thus, finding the asymptotics of the integral $\int_{M \in O(n)} M_{i_1 j_1} \dots M_{i_{2r} j_{2r}} dM$ is equivalent to finding the (admissible) pairings \mathbf{m} and \mathbf{n} that maximize the number of components in the graph. [4] also contains a useful table of values for the orthogonal Weingarten functions of small dimension.

For our purposes, it will be simpler to use the stronger Lyapunov's condition in place of the Lindeberg condition in McLeish's theorem. For $\delta > 0$, Lyapunov's condition says that

$$\lim_{n \rightarrow \infty} \frac{1}{s_n^{2+\delta}} \sum_{i=1}^n \mathbb{E}[|X_{n,i}|^{2+\delta}] = 0. \quad (2.1)$$

We apply this now to show

Lemma 2.2.

$$\sqrt{n} \sum_{i=1}^n M_{a,i} M_{b,i+1} \xrightarrow{d} \mathcal{N}(0, 1)$$

Proof. Let $X_{n,i} = \sqrt{n} M_{a,i} M_{b,i+1}$. By sign symmetry, $\mathbb{E}[X_{n,i}] = 0$. Note that the random vector $(X_{n,1}, X_{n,2}, \dots, X_{n,i-1})$ is only a function of $(M_{a,1}, M_{a,2}, \dots, M_{a,i-1}, M_{b,1}, M_{b,2}, \dots, M_{b,i})$. Since $M_{b,i+1}$ does not share a column with any of these matrix entries, its sign can be reversed independently while preserving Haar measure. Thus,

$$(X_{n,1}, X_{n,2}, \dots, X_{n,i-1}, M_{b,i+1}) \stackrel{d}{=} (X_{n,1}, X_{n,2}, \dots, X_{n,i-1}, -M_{b,i+1})$$

and therefore

$$\mathbb{E}(X_{n,i} | \mathcal{F}_{n,i-1}) = \sqrt{n} \mathbb{E}(M_{a,i} M_{b,i+1} | \sigma(X_{n,1}, X_{n,2}, \dots, X_{n,i-1})) = 0.$$

This shows that $X_{n,i}$ is a martingale difference array. Let's now verify the conditions in McLeish's theorem.

The variance $\sigma_{n,i}^2$ is $\frac{1}{n+2}$ when $a = b$ and $\frac{n+1}{(n-1)(n+2)}$ when $a \neq b$ (see e.g. [20, chapt. 4]). Thus, $\lim_{n \rightarrow \infty} \sum_{i=1}^n \sigma_{n,i}^2 = 1$ and the array is already normalized.

By the row invariance of Haar measure, all the random variables $X_{n,i}$ are identically distributed. Thus, Lyapunov's condition for $\delta = 2$ simply states that $\lim_{n \rightarrow \infty} n \mathbb{E}[X_{n,i}^4] = 0$. By Lemma 2.1, $n^2 \mathbb{E}[X_{n,i}^4] = n^4 \mathbb{E}[M_{a,i}^4 M_{b,i+1}^4] = O(1)$ and Lyapunov's condition is satisfied.

The last condition to show is that $\limsup_{n \rightarrow \infty} \sum_{i \neq j} \mathbb{E}[X_{n,i}^2 X_{n,j}^2] \leq 1$. We split this into two sums: $\sum_{i \neq j} \mathbb{E}[X_{n,i}^2 X_{n,j}^2] = \sum_{|i-j| > 1} \mathbb{E}[X_{n,i}^2 X_{n,j}^2] + \sum_{|i-j|=1} \mathbb{E}[X_{n,i}^2 X_{n,j}^2]$. Note that the random variables $X_{n,i}^2 X_{n,j}^2$ such that $|i-j| > 1$ are all identically distributed, and similarly for $|i-j|=1$. The limit of the first sum is

$$\lim_{n \rightarrow \infty} \sum_{|i-j| > 1} \mathbb{E}[X_{n,i}^2 X_{n,j}^2] = \lim_{n \rightarrow \infty} \mathbb{E}[n^2 X_{n,1}^2 X_{n,3}^2] = \lim_{n \rightarrow \infty} \mathbb{E}[n^4 M_{a,1}^2 M_{b,2}^2 M_{a,3}^2 M_{b,4}^2] = 1.$$

The asymptotic of the second sum is

$$\sum_{|i-j|=1} \mathbb{E}[X_{n,i}^2 X_{n,j}^2] = O(1/n)$$

and therefore

$$\limsup_{n \rightarrow \infty} \sum_{i \neq j} \mathbb{E}[X_{n,i}^2 X_{n,j}^2] = 1.$$

□

We've shown that if C_n is the matrix corresponding to the n -cycle $(1, 2, 3, \dots, n)$ then each matrix entry $(MC_n M^T)_{ab}$ scaled by \sqrt{n} converges to the standard normal distribution. It turns out that C_n is in fact a fairly generic permutation in \mathfrak{S}_n . The following lemma makes this precise.

Lemma 2.3. *Let M be an $n \times n$ Haar distributed orthogonal matrix and Z be a $N(0, 1)$ distributed random variable. Then*

$$\lim_{n \rightarrow \infty} \max_{Q_n} |\mathbb{P}\{\sqrt{n}(MQ_n M^T)_{ab} \leq x\} - \mathbb{P}\{Z \leq x\}| = 0$$

where the maximum is taken over the set of all $n \times n$ permutation matrices, Q_n , such that the number of cycles in the corresponding permutation is at most $2 \log n$.

Proof. Let $\delta > 0$ and let Q_n be a permutation matrix whose corresponding permutation $\sigma \in \mathfrak{S}_n$ has $k_n \leq 2 \log n$ cycles. For ease of notation, we set $X_n = \sqrt{n}(MQ_n M^T)_{ab}$ and $Y_n = \sqrt{n}(MC_n M^T)_{ab}$. Since the distribution of X_n only depends on the cycle type of σ , we can assume $\sigma = (1, 2, 3, \dots, n_1)(n_1 + 1, n_1 + 2, \dots, n_2) \dots (n_{k_n-1} + 1, n_{k_n-1} + 2, \dots, n)$ when decomposed into cycles. Then

$$X_n - Y_n = \sum_{i=1}^n M_{a,i} M_{b,\sigma(i)} - \sum_{i=1}^n M_{a,i} M_{b,i+1} = \sum_{i=1}^{2k_n} W_i$$

where $W_i \stackrel{d}{=} \sqrt{n} M_{a,c_i} M_{b,d_i}$ for some indices c_i and d_i . For sufficiently large n , $\mathbb{E}[W_i^2] < C/n$ for some constant C and by Cauchy-Schwarz,

$$\mathbb{E}[(X_n - Y_n)^2] < 4(k_n)^2(C/n).$$

By Chebyshev's inequality,

$$\mathbb{P}\{|X_n - Y_n| > \varepsilon\} \leq \varepsilon^{-2} \mathbb{E}[(X_n - Y_n)^2] \leq \varepsilon^{-2} \frac{16C \log^2 n}{n}.$$

We also have the two inequalities

$$\begin{aligned} \mathbb{P}\{X_n \leq x\} &\leq \mathbb{P}\{Y_n \leq x + \varepsilon\} + \mathbb{P}\{|X_n - Y_n| > \varepsilon\} \\ \mathbb{P}\{Y_n \leq x - \varepsilon\} &\leq \mathbb{P}\{X_n \leq x\} + \mathbb{P}\{|X_n - Y_n| > \varepsilon\} \end{aligned}$$

Putting them together, we have

$$\mathbb{P}\{Y_n \leq x - \varepsilon\} - \mathbb{P}\{|X_n - Y_n| > \varepsilon\} \leq \mathbb{P}\{X_n \leq x\} \leq \mathbb{P}\{Y_n \leq x + \varepsilon\} + \mathbb{P}\{|X_n - Y_n| > \varepsilon\}$$

Thus,

$$\begin{aligned} |\mathbb{P}\{X_n \leq x\} - \mathbb{P}\{Y_n \leq x\}| &\leq \mathbb{P}\{Y_n \leq x + \varepsilon\} - \mathbb{P}\{Y_n \leq x - \varepsilon\} + 2\mathbb{P}\{|X_n - Y_n| > \varepsilon\} \\ &\leq \mathbb{P}\{Y_n \leq x + \varepsilon\} - \mathbb{P}\{Y_n \leq x - \varepsilon\} + \varepsilon^{-2} \frac{32C \log^2 n}{n} \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} |\mathbb{P}\{Y_n \leq x + \varepsilon\} - \mathbb{P}\{Y_n \leq x - \varepsilon\}| &\leq |\mathbb{P}\{Y_n \leq x + \varepsilon\} - \mathbb{P}\{Z \leq x + \varepsilon\}| \\ &\quad + |\mathbb{P}\{Z \leq x + \varepsilon\} - \mathbb{P}\{Z \leq x - \varepsilon\}| + |\mathbb{P}\{Y_n \leq x - \varepsilon\} - \mathbb{P}\{Z \leq x - \varepsilon\}| \end{aligned}$$

Also,

$$|\mathbb{P}\{X_n \leq x\} - \mathbb{P}\{Z \leq x\}| \leq |\mathbb{P}\{X_n \leq x\} - \mathbb{P}\{Y_n \leq x\}| + |\mathbb{P}\{Y_n \leq x\} - \mathbb{P}\{Z \leq x\}|.$$

Since Y_n converges weakly to Z , $|\mathbb{P}\{Y_n \leq x\} - \mathbb{P}\{Z \leq x\}| < \delta/2$ for sufficiently large n . Choose small enough ε such that $\mathbb{P}\{Z \leq x + \varepsilon\} - \mathbb{P}\{Z \leq x - \varepsilon\} < \delta/10$. For large enough n , the three quantities $|\mathbb{P}\{Y_n \leq x + \varepsilon\} - \mathbb{P}\{Z \leq x + \varepsilon\}|$, $|\mathbb{P}\{Y_n \leq x - \varepsilon\} - \mathbb{P}\{Z \leq x - \varepsilon\}|$, and $\varepsilon^{-2} \frac{32C \log^2 n}{n}$ are all smaller than $\delta/10$. Then $|\mathbb{P}\{X_n \leq x\} - \mathbb{P}\{Y_n \leq x\}| < \delta/2$. Since Q_n was arbitrary, this proves the lemma. \square

Let $K_n(\sigma)$ denote the number of cycles in a random permutation $\sigma \in \mathfrak{S}_n$. The classical Goncharov theorem (see e.g. [12, p. 116]) states that $\frac{K_n - \log n}{\sqrt{\log n}} \xrightarrow{d} \mathcal{N}(0, 1)$.

This implies in particular that $\mathbb{P}\{K_n > 2 \log n\} \rightarrow 0$.

Now we are ready to prove:

Theorem 2.2. *Let M be a Haar distributed orthogonal $n \times n$ matrix and P a random $n \times n$ permutation matrix independent of M . Then*

$$\sqrt{n}(MPM^T)_{ab} \xrightarrow{d} \mathcal{N}(0, 1)$$

Proof. Averaging over all permutation matrices,

$$\mathbb{P}\{\sqrt{n}(MPM^T)_{ab} \leq x\} = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \mathbb{P}\{\sqrt{n}(MP_\sigma M^T)_{ab} \leq x\}$$

For any $\delta > 0$ and sufficiently large n , Lemma 2.3 gives

$$\begin{aligned} &|\mathbb{P}\{\sqrt{n}(MPM^T)_{ab} \leq x\} - \mathbb{P}\{Z \leq x\}| \\ &\leq \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} |\mathbb{P}\{\sqrt{n}(MP_\sigma M^T)_{ab} \leq x\} - \mathbb{P}\{Z \leq x\}| \\ &\leq \delta/2 + \mathbb{P}\{K_n > 2 \log n\} \\ &< \delta \end{aligned}$$

\square

3. LIMITING JOINT DISTRIBUTIONS

We can generalize Theorem 2.2 to obtain limiting joint distributions of the entries of MPM^T . By the Cramér-Wold Device, this is equivalent to determining the limiting distribution of arbitrary linear combinations of the matrix entries. In other words, for a sequence of $n \times n$ real matrices A_n , we wish to determine the asymptotic behavior of $\text{Tr}(A_n MPM^T)$. Theorem 2.2 says that this trace converges to a standard normal when the (a, b) entry of A_n is \sqrt{n} and all the other entries are 0.

We will now consider more general coefficient matrices A_n . For convenience, the subscript will often be dropped. Normalize A so that $\text{Tr}(AA^T) = n$.

Remark 3.1. Recall that the limiting normality of $Tr(AM)$ was proven in [6]. A simple but key observation in the proof is that we can reduce to the case of diagonal A by using the singular value decomposition. Writing $A = UDV^T$, we get $Tr(AM) = Tr(UDV^TM) = Tr(DV^TMU)$. This is equal in distribution to $Tr(DM)$ by left and right Haar invariance.

Unfortunately, we cannot reduce to the case of diagonal matrices in our situation since

$$\begin{aligned} Tr(AMPMT^T) &= Tr(UDV^T MPM^T) = Tr(DV^T MPM^T U) \\ &= Tr(D(V^T M)P(U^T M)^T) \end{aligned}$$

and in general $U \neq V$. If A is normal, e.g. symmetric or orthogonal, then $U = V$ and it would be possible to reduce to diagonal A . However, we will not make this assumption.

As before, let C_n denote the $n \times n$ permutation matrix corresponding to the n -cycle $(1, 2, \dots, n)$. Then

$$Tr(AMC_n M^T) = \sum_{i=1}^n \sum_{j=1}^n A_{ij} (MC_n M^T)_{ji} = \sum_{k=1}^n X_{n,k}$$

where $X_{n,k} = \sum_{i=1}^n \sum_{j=1}^n A_{ij} M_{j,k} M_{i,k+1}$. It is easy to see that $X_{n,k}$ is a martingale difference array. Squaring $X_{n,k}$ and expanding, we get

$$\begin{aligned} X_{n,k}^2 &= \sum_{i=1}^n \sum_{j=1}^n A_{ij}^2 M_{j,k}^2 M_{i,k+1}^2 \\ &+ \sum_{i=1}^n \sum_{j=1}^n \sum_{m \neq j} A_{ij} A_{im} M_{j,k} M_{m,k} M_{i,k+1}^2 + \sum_{i=1}^n \sum_{j=1}^n \sum_{l \neq i} A_{ij} A_{lj} M_{j,k}^2 M_{i,k+1} M_{l,k+1} \\ &+ \sum_{i=1}^n \sum_{j=1}^n \sum_{l \neq i} \sum_{m \neq j} A_{ij} A_{lm} M_{j,k} M_{i,k+1} M_{m,k} M_{l,k+1} \end{aligned}$$

By Remark 2.1, the mixed moments are

$$\begin{aligned} \mathbb{E}[M_{j,k}^2 M_{i,k+1}^2] &= 1/n^2 + O(1/n^3) \\ \mathbb{E}[M_{j,k}^2 M_{i,k+1} M_{l,k+1}] &= 0 \\ \mathbb{E}[M_{j,k} M_{i,k+1} M_{m,k} M_{l,k+1}] &= \begin{cases} \frac{-1}{(n-1)n(n+2)} & \text{if } j=i, m=l \text{ or } j=l, i=m \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Therefore the variance is

$$\mathbb{E}[X_{n,k}^2] = \left(\frac{1}{n^2} + O\left(\frac{1}{n^3}\right) \right) \sum_{i=1}^n \sum_{j=1}^n A_{ij}^2 - \frac{1}{(n-1)n(n+2)} \sum_{i \neq j} (A_{ij} A_{ji} + A_{ii} A_{jj}) \quad (3.1)$$

Recall we have the constraint $\sum A_{ij}^2 = n$. It is clear that $\sum_{i \neq j} A_{ij} A_{ji}$ is maximized when $A_{ij} = A_{ji}$ and minimized when $A_{ij} = -A_{ji}$. In any case, the magnitude of

the sum is at most n . By the method of Lagrange multipliers, $\sum_{i \neq j} A_{ii} A_{jj}$ achieves a maximum value of $n^2 - n$ when A is the identity matrix and a minimum value of $-n$ when $\sum A_{ii} = 0$. Without any further restrictions, the sum $\sum_{i \neq j} A_{ii} A_{jj}$ can fluctuate wildly. For there to be any hope of a limiting distribution for $\text{Tr}(AMC_n M^T)$, we need $\lim_{n \rightarrow \infty} \sum_{i \neq j} \frac{A_{ii} A_{jj}}{n^2} = c$ for some constant $0 \leq c \leq 1$. Then we have the following:

Theorem 3.1. *Let A_n be a sequence of $n \times n$ matrices such that $\text{Tr}(A^T A) = n$. As usual M denotes an $n \times n$ Haar distributed orthogonal matrix and C_n the permutation matrix $(1, 2, \dots, n)$. If $\lim_{n \rightarrow \infty} \sum_{i \neq j} \frac{A_{ii} A_{jj}}{n^2} = c$, then $\text{Tr}(AMC_n M^T) \xrightarrow{d} \mathcal{N}(0, 1 - c)$.*

Proof. From equation (3.1), we see that the limiting variance of $\text{Tr}(AMC_n M^T)$ is given by

$$\lim_{n \rightarrow \infty} \sum_{k=1}^n \sigma_{n,k}^2 = \lim_{n \rightarrow \infty} n \mathbb{E} X_{n,1}^2 = 1 - c$$

Next, we show that Lyapunov's condition for the martingale difference array $X_{n,k}$ holds with $\delta = 2$, i.e. $\lim_{n \rightarrow \infty} n \mathbb{E}[X_{n,k}^4] = 0$.

When expanded out, $X_{n,k}^4$ contains terms of the form $\prod_{p=1}^4 A_{i_p j_p} M_{j_p, k} M_{i_p, k+1}$.

Recall the only terms with non-zero expectation are those with an even number of matrix entries M_{ij} in each row and column. Working through the possibilities, we get (using the shorthand $i \neq j \neq l \neq m$ to mean i, j, l, m all distinct and \ll to denote inequality up to an absolute constant as $n \rightarrow \infty$)

$$\begin{aligned} \mathbb{E}[X_{n,k}^4] &\ll \sum_{i \neq j \neq l \neq m} (A_{ii} A_{jj} A_{ll} A_{mm} + A_{ij} A_{ji} A_{ll} A_{mm} + A_{ij} A_{ji} A_{lm} A_{ml} \\ &\quad + A_{ij} A_{jl} A_{li} A_{mm} + A_{ij} A_{jl} A_{lm} A_{mi}) \mathbb{E}[M_{i,k} M_{i,k+1} M_{j,k} M_{j,k+1} M_{l,k} M_{l,k+1} M_{m,k} M_{m,k+1}] \\ &\quad + \sum_{\substack{i \neq j \neq l \\ i \neq j \neq m}} (A_{ii} A_{jj} A_{ll}^2 + A_{ii} A_{jl} A_{lj} A_{ll} + A_{ij} A_{ji} A_{ll}^2 + A_{ij} A_{jl} A_{li} A_{ll} \\ &\quad + A_{il} A_{jl} A_{li} A_{lj}) \mathbb{E}[M_{i,k} M_{i,k+1} M_{j,k} M_{j,k+1} M_{l,k}^2 M_{m,k+1}^2] \\ &\quad + \sum_{i \neq j \neq l} (A_{ii} A_{jj} A_{il}^2 + A_{ii} A_{ij} A_{il} A_{jl} + A_{ji} A_{ij} A_{il}^2) \mathbb{E}[M_{i,k} M_{i,k+1}^3 M_{j,k} M_{j,k+1} M_{l,k}^2] \\ &\quad + \sum_{i \neq j} (A_{ii} A_{jj}^3 + A_{ij} A_{ji} A_{jj}^2) \mathbb{E}[M_{i,k} M_{i,k+1} M_{j,k}^3 M_{j,k+1}^3] \\ &\quad + \sum_{i \neq j} (A_{ii} A_{jj} A_{ij}^2 + A_{ji} A_{ij}^3) \mathbb{E}[M_{i,k} M_{i,k+1}^3 M_{j,k}^3 M_{j,k+1}] \\ &\quad + \sum_{\substack{i \neq l \\ j \neq m}} (A_{ji}^2 A_{ml}^2 + A_{ji} A_{mi} A_{jl} A_{ml}) \mathbb{E}[M_{i,k}^2 M_{j,k+1}^2 M_{l,k}^2 M_{m,k+1}^2] \\ &\quad + \sum_{\substack{j \\ i \neq l}} (A_{ji}^2 A_{jl}^2) \mathbb{E}[M_{i,k}^2 M_{j,k+1}^4 M_{l,k}^2] \end{aligned}$$

$$+ \sum_{i,j} (A_{ji}^4) \mathbb{E}[M_{i,k}^4 M_{j,k+1}^4]$$

By the discussion in Remark 2.1, it is easy to compute

$$\begin{aligned} \mathbb{E}[M_{i,k} M_{i,k+1} M_{j,k} M_{j,k+1} M_{l,k} M_{l,k+1} M_{m,k} M_{m,k+1}] &= O(1/n^6) \\ \mathbb{E}[M_{i,k} M_{i,k+1} M_{j,k} M_{j,k+1} M_{l,k}^2 M_{m,k+1}^2] &= O(1/n^5) \\ \mathbb{E}[M_{i,k} M_{i,k+1}^3 M_{j,k} M_{j,k+1} M_{l,k}^2] &= O(1/n^5) \\ \mathbb{E}[M_{i,k} M_{i,k+1} M_{j,k}^3 M_{j,k+1}^3] &= O(1/n^5) \\ \mathbb{E}[M_{i,k} M_{i,k+1}^3 M_{j,k}^3 M_{j,k+1}] &= O(1/n^5) \\ \mathbb{E}[M_{i,k}^2 M_{j,k+1}^2 M_{l,k}^2 M_{m,k+1}^2] &= O(1/n^4) \\ \mathbb{E}[M_{i,k}^2 M_{j,k+1}^4 M_{l,k}^2] &= O(1/n^4) \\ \mathbb{E}[M_{i,k}^4 M_{j,k+1}^4] &= O(1/n^4) \end{aligned}$$

The sums involving the matrix entries of A are straightforward to bound. For example, we have

$$\sum_{\substack{i \neq l \\ j \neq m}} A_{ij} A_{im} A_{lj} A_{lm} \leq \sum_{i,j,l,m} (A_{ij}^2 A_{lm}^2 + A_{im}^2 A_{lj}^2) / 2 \leq n^2$$

Going through all the terms, we see that $\mathbb{E}[X_{n,k}^4] = O(1/n^2)$ and Lyapunov's condition is satisfied.

The last condition of McLeish's Theorem to show is

$$\limsup_{n \rightarrow \infty} \sum_{k \neq l} \mathbb{E}[X_{n,k}^2 X_{n,l}^2] = (1 - c)^2.$$

(Note we didn't normalize the array $X_{n,k}$). By Cauchy-Schwarz,

$$\mathbb{E}[X_{n,k}^2 X_{n,l}^2] \leq \mathbb{E}[X_{n,k}^4] = O(1/n^2).$$

Thus, $\lim_{n \rightarrow \infty} \sum_{|l-k|=1} \mathbb{E}[X_{n,k}^2 X_{n,l}^2] = 0$. Therefore, we can assume $|l - k| > 1$.

Consider the expectation of

$$X_{n,k}^2 X_{n,l}^2 = \left(\sum_{a=1}^n \sum_{b=1}^n A_{ab} M_{b,k} M_{a,k+1} \right)^2 \left(\sum_{c=1}^n \sum_{d=1}^n A_{cd} M_{d,l} M_{c,l+1} \right)^2$$

Again, going through all the possibilities, we have the following types of terms:

- (1) $\sum_{\substack{a \neq b \\ c \neq d}} (A_{aa} A_{bb} A_{cc} A_{dd} + A_{ab} A_{ba} A_{cd} A_{dc} + A_{aa} A_{bb} A_{cd} A_{dc}) \mathbb{E}[M_{a,k} M_{a,k+1} M_{b,k} M_{b,k+1} M_{c,l} M_{c,l+1} M_{d,l} M_{d,l+1}]$
- (2) $\sum_{\substack{a \neq b \\ c \neq d}} (A_{ca} A_{db} A_{ca} A_{db} + A_{ca} A_{db} A_{da} A_{cb}) \mathbb{E}[M_{a,k} M_{c,k+1} M_{b,k} M_{d,k+1} M_{a,l} M_{c,l+1} M_{b,l} M_{d,l+1}]$
- (3) $\sum_{a,d \neq b,c} (A_{aa} A_{cb} A_{cb} A_{dd} + A_{aa} A_{cb} A_{cd} A_{db} + A_{ca} A_{ab} A_{cd} A_{db}) \mathbb{E}[M_{a,k} M_{a,k+1} M_{b,k} M_{c,k+1} M_{b,l} M_{c,l+1} M_{d,l} M_{d,l+1}]$

$$\begin{aligned}
(4) \quad & \sum_{a,d \neq b,c} (A_{ca}A_{db}A_{ba}A_{dc} + A_{ca}A_{db}A_{da}A_{bc} + A_{da}A_{cb}A_{ba}A_{dc} \\
& + A_{da}A_{cb}A_{da}A_{bc}) \mathbb{E}[M_{a,k}M_{c,k+1}M_{b,k}M_{d,k+1}M_{a,l}M_{b,l+1}M_{c,l}M_{d,l+1}] \\
(5) \quad & \sum_{\substack{d \\ a \neq b \neq c}} (A_{aa}A_{cb}A_{db}A_{dc} + A_{ca}A_{ab}A_{db}A_{dc}) \mathbb{E}[M_{a,k}M_{a,k+1}M_{b,k}M_{c,k+1}M_{b,l}M_{c,l}M_{d,l+1}^2] \\
(6) \quad & \sum_{\substack{c,d \\ a \neq b}} (A_{aa}A_{bb}A_{dc}^2 + A_{ab}A_{ba}A_{dc}^2) \mathbb{E}[M_{a,k}M_{a,k+1}M_{b,k}M_{b,k+1}M_{c,l}^2M_{d,l+1}^2] \\
(7) \quad & \sum_{\substack{c,d \\ a \neq b}} (A_{ca}A_{cb}A_{da}A_{db}) \mathbb{E}[M_{a,k}M_{b,k}M_{c,k+1}^2M_{a,l}M_{b,l}M_{d,l+1}^2] \\
(8) \quad & \sum_{a,b,c,d} (A_{ba}^2A_{dc}^2) \mathbb{E}[M_{a,k}^2M_{b,k+1}^2M_{c,l}^2M_{d,l+1}^2]
\end{aligned}$$

The orthogonal integrals (using Remark 2.1) are given by

$$\begin{aligned}
\mathbb{E}[M_{a,k}M_{a,k+1}M_{b,k}M_{b,k+1}M_{c,l}M_{c,l+1}M_{d,l}M_{d,l+1}] &= O(1/n^6) \\
\mathbb{E}[M_{a,k}M_{c,k+1}M_{b,k}M_{d,k+1}M_{a,l}M_{c,l+1}M_{b,l}M_{d,l+1}] &= O(1/n^6) \\
\mathbb{E}[M_{a,k}M_{a,k+1}M_{b,k}M_{c,k+1}M_{b,l}M_{c,l+1}M_{d,l}M_{d,l+1}] &= O(1/n^7) \\
\mathbb{E}[M_{a,k}M_{c,k+1}M_{b,k}M_{d,k+1}M_{a,l}M_{b,l+1}M_{c,l}M_{d,l+1}] &= O(1/n^7) \\
\mathbb{E}[M_{a,k}M_{a,k+1}M_{b,k}M_{c,k+1}M_{b,l}M_{c,l}M_{d,l+1}^2] &= O(1/n^6) \\
\mathbb{E}[M_{a,k}M_{a,k+1}M_{b,k}M_{b,k+1}M_{c,l}^2M_{d,l+1}^2] &= O(1/n^5) \\
\mathbb{E}[M_{a,k}M_{b,k}M_{c,k+1}^2M_{a,l}M_{b,l}M_{d,l+1}^2] &= O(1/n^5) \\
\mathbb{E}[M_{a,k}^2M_{b,k+1}^2M_{c,l}^2M_{d,l+1}^2] &= O(1/n^4)
\end{aligned}$$

Going through the 8 types of terms in the expansion of $\mathbb{E}[X_{n,k}^2X_{n,l}^2]$, we see that the 2nd, 3rd, 4th, 5th, and 7th terms are all $O(1/n^3)$. The 1st is $c^2/n^2 + O(1/n^3)$, the 6th is $-c/n^2 + O(1/n^3)$, and the 8th is $1/n^2 + O(1/n^3)$. Note that in the expansion, we actually get two copies of terms of the 6th type since k and l can be switched.

Thus, $\mathbb{E}[X_{n,k}^2X_{n,l}^2] = (1 - 2c + c^2)/n^2 + O(1/n^3)$. When $|l - k| > 1$, $X_{n,k}^2X_{n,l}^2$ are all identically distributed. This proves that $\limsup_{n \rightarrow \infty} \sum_{k \neq l} \mathbb{E}[X_{n,k}^2X_{n,l}^2] = (1 - c)^2$. \square

Theorem 3.1 shows that $Tr(AMC_nM^T) \xrightarrow{d} \mathcal{N}(0, 1 - c)$ where $C_n = (1, 2, \dots, n)$. Unlike in Section 2, C_n is no longer a generic permutation. The asymptotic distribution of $Tr(AMQM^T)$ will depend on the number of fixed points of the permutation Q . The following lemma generalizes Lemma 2.3 from the previous section.

Lemma 3.1. *Let A_n be a sequence of $n \times n$ matrices such that $Tr(A^T A) = n$ and*

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{A_{ii}}{n} = s$$

for some constant s . Let M be an $n \times n$ Haar distributed orthogonal matrix and

$$Z \stackrel{d}{=} \mathcal{N}(0, 1 - s^2).$$

Then for every non-negative integer f ,

$$\lim_{n \rightarrow \infty} \max_{Q_n} |\mathbb{P}\{Tr(AMQ_nM^T) \leq x\} - \mathbb{P}\{Z + fs \leq x\}| = 0$$

where the maximum is taken over the set of all $n \times n$ permutation matrices, Q_n , with f fixed points and at most $2 \log n$ cycles.

Proof. First, note that $\lim_{n \rightarrow \infty} \sum_{i \neq j} \frac{A_{ii}A_{jj}}{n^2} = s^2$.

Let Q_n be a permutation matrix with f fixed points and $k_n \leq 2 \log n$ cycles. As before, we can assume the permutation has the form $(1, 2, \dots, n_1)(n_1 + 1, n_1 + 2, \dots, n_2) \dots (n_{k_n-1} + 1, n_{k_n-1} + 2, \dots, n)$. Set $X_n = Tr(AMQ_nM^T)$ and $Y_n = Tr(AMC_nM^T)$. Then the difference is

$$X_n - Y_n = \sum_{m=1}^{2k_n-f} W_m + \sum_{m=1}^f (V_{m,1} + V_{m,2})$$

where

$$W_m \stackrel{d}{=} \sum_{i=1}^n \sum_{j=1}^n A_{ij} M_{j,k} M_{i,k+1}$$

$$V_{m,1} \stackrel{d}{=} \sum_{i \neq j} A_{ij} M_{j,k} M_{i,k}$$

and

$$V_{m,2} \stackrel{d}{=} \sum_{i=1}^n A_{ii} M_{i,k}^2.$$

Since $\mathbb{E} \left[\left(\sum_{i=1}^n A_{ii} (M_{i,k}^2 - 1/n) \right)^2 \right] = O(1/n)$, we have the convergence

$$V_{m,2} \xrightarrow{L^2} s$$

and therefore

$$\sum_{m=1}^f V_{m,2} \xrightarrow{d} fs.$$

Let $Y'_n = Y_n + \sum_{m=1}^f V_{m,2}$. For large n , we have the asymptotics

$$n\mathbb{E}[V_{m,1}^2] = n \sum_{i \neq j} (A_{ij}^2 + A_{ij}A_{ji}) \mathbb{E}[M_{j,k}^2 M_{i,k}^2] = O(1)$$

and

$$n\mathbb{E}[W_m^2] = O(1).$$

Therefore, by Cauchy-Schwarz,

$$\mathbb{E}[(X_n - Y'_n)^2] \leq (2k_n)^2 (C/n)$$

for sufficiently large n for some constant C and therefore by Chebyshev,

$$\mathbb{P}\{|X_n - Y'_n| > \varepsilon\} \leq \varepsilon^{-2} (2k_n)^2 (C/n) \leq \varepsilon^{-2} \frac{16C \log^2 n}{n}.$$

From the proof of Lemma 2.3,

$$\begin{aligned} |\mathbb{P}\{X_n \leq x\} - \mathbb{P}\{Y'_n \leq x\}| &\leq \mathbb{P}\{Y'_n \leq x + \varepsilon\} - \mathbb{P}\{Y'_n \leq x - \varepsilon\} + 2\mathbb{P}\{|X_n - Y'_n| > \varepsilon\} \\ &\leq \mathbb{P}\{Y'_n \leq x + \varepsilon\} - \mathbb{P}\{Y'_n \leq x - \varepsilon\} + \varepsilon^{-2} \frac{32C \log^2 n}{n} \end{aligned}$$

By the triangle inequality,

$$\begin{aligned} &\mathbb{P}\{Y'_n \leq x + \varepsilon\} - \mathbb{P}\{Y'_n \leq x - \varepsilon\} \leq |\mathbb{P}\{Y'_n \leq x + \varepsilon\} - \mathbb{P}\{Z + fs \leq x + \varepsilon\}| \\ &+ \mathbb{P}\{Z + fs \leq x + \varepsilon\} - \mathbb{P}\{Z + fs \leq x - \varepsilon\} + |\mathbb{P}\{Y'_n \leq x - \varepsilon\} - \mathbb{P}\{Z + fs \leq x - \varepsilon\}| \end{aligned}$$

By Theorem 3.1 (and the converging together lemma), Y'_n converges weakly to $Z + fs$. Thus,

$$|\mathbb{P}\{Y'_n \leq x\} - \mathbb{P}\{Z + fs \leq x\}| < \delta/2$$

for $\delta > 0$ and sufficiently large n . Choose ε small enough such that $\mathbb{P}\{Z + fs \leq x + \varepsilon\} - \mathbb{P}\{Z + fs \leq x - \varepsilon\} < \delta/10$. For large enough n , the three quantities $|\mathbb{P}\{Y'_n \leq x + \varepsilon\} - \mathbb{P}\{Z + fs \leq x + \varepsilon\}|$, $|\mathbb{P}\{Y'_n \leq x - \varepsilon\} - \mathbb{P}\{Z + fs \leq x - \varepsilon\}|$, and $\varepsilon^{-2} \frac{32C \log^2 n}{n}$ are all bounded by $\delta/10$. Then $|\mathbb{P}\{X_n \leq x\} - \mathbb{P}\{Y'_n \leq x\}| < \delta/2$ and we have $|\mathbb{P}\{X_n \leq x\} - \mathbb{P}\{Z + fs \leq x\}| < \delta$ for sufficiently large n . Since Q_n was arbitrary, this proves the lemma. \square

Putting everything together, we have the main result.

Theorem 3.2. *Let A_n, M, Z be as defined in Lemma 3.1. Let P be an $n \times n$ random permutation matrix and let $Y \stackrel{d}{=} \text{Pois}(1)$ chosen independently from Z . Then in the limit of large n ,*

$$\text{Tr}(AMPMT^T) \xrightarrow{d} Z + sY.$$

Proof. Let $[f]_n$ be the subset of permutations in \mathfrak{S}_n with f fixed points and let F_n denote the number of fixed points of a random permutation in \mathfrak{S}_n . Recall that $F_n \xrightarrow{d} Y$.

Let $\delta > 0$. Choose a large integer F such that $\mathbb{P}(Y > F) < \delta/5$. By Lemma 3.1, for sufficiently large n ,

$$\begin{aligned} &|\mathbb{P}\{\text{Tr}(AMPMT^T) \leq x\} - \mathbb{P}\{Z + sY \leq x\}| \\ &= \left| \frac{1}{n!} \sum_{f=1}^{\infty} \sum_{\sigma \in [f]_n} \mathbb{P}\{\text{Tr}(AMP_{\sigma}M^T) \leq x\} - \sum_{f=1}^{\infty} \mathbb{P}\{Y = f\} \mathbb{P}\{Z + fs \leq x\} \right| \\ &\leq \mathbb{P}\{F_n > F\} + \mathbb{P}\{Y > F\} + \sum_{f=1}^F \left| \frac{1}{n!} \sum_{\sigma \in [f]_n} \mathbb{P}\{\text{Tr}(AMP_{\sigma}M^T) \leq x\} \right. \\ &\quad \left. - \mathbb{P}\{Y = f\} \mathbb{P}\{Z + fs \leq x\} \right| \\ &\leq \frac{\delta}{2} + \sum_{f=1}^F \left(\left| \frac{1}{n!} \sum_{\sigma \in [f]_n} (\mathbb{P}\{\text{Tr}(AMP_{\sigma}M^T) \leq x\} - \mathbb{P}\{Z + fs \leq x\}) \right| \right. \\ &\quad \left. + |\mathbb{P}\{F_n = f\} - \mathbb{P}\{Y = f\}| \right) \\ &\leq \frac{2\delta}{3} + \frac{1}{n!} \sum_{f=1}^F \sum_{\sigma \in [f]_n} \left| \mathbb{P}\{\text{Tr}(AMP_{\sigma}M^T) \leq x\} - \mathbb{P}\{Z + fs \leq x\} \right| \end{aligned}$$

$< \delta$

□

The following corollaries illustrate a few special cases of this theorem.

Corollary 3.1. *Let k be a fixed positive integer. Let M be an $n \times n$ Haar-distributed orthogonal matrix and P be a random $n \times n$ permutation matrix. Then the joint distribution of k entries of the random matrix MPM^T normalized by \sqrt{n} is asymptotically jointly i.i.d standard normal as $n \rightarrow \infty$.*

Corollary 3.2. *Let A_n be a sequence of diagonal $n \times n$ matrices such that $\text{Tr}(AA^T) = n$ and $A_{ii} = \sqrt{1/\alpha}$ for $1 \leq i \leq \alpha n$ for some parameter $0 < \alpha \leq 1$. Let $Z \stackrel{d}{=} \mathcal{N}(0, 1 - \alpha)$ and $Y \stackrel{d}{=} \text{Pois}(1)$ be independent random variables. Then*

$$\text{Tr}(AMPMT) \stackrel{d}{\rightarrow} Z + \sqrt{\alpha}Y.$$

Remark 3.2. The result in Theorem 3.2 is for the defining representation of the symmetric group. It would be of interest to see if similar distributional results for linear combinations of the matrix entries also hold for higher dimensional representations of the symmetric group.

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REFERENCES

- [1] G. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*. Vol. 118. Cambridge Studies in Advanced Mathematics. Cambridge: Cambridge University Press, 2010.
- [2] E. Borel. “Sur les principes de la theorie cinetique des gazs”. In: *Annales de l’ecole normale sup.* 23 (1906), pp. 9–32.
- [3] S. Chatterjee and E. Meckes. “Multivariate normal approximation using exchangeable pairs”. In: *ALEA Lat. Am. J. Probab. Math. Stat.* 4 (2008), pp. 257–283.
- [4] B. Collins and S. Matsumoto. “On some properties of orthogonal Weingarten functions”. In: *J. Math. Phys.* 50.113516 (2009).
- [5] B. Collins and P. Sniady. “Integration with respect to the Haar measure on unitary, orthogonal and symplectic group”. In: *Commun. Math. Phys.* 264.3 (2006), pp. 773–795.
- [6] A. D’Aristotile, P. Diaconis, and C. Newman. “Brownian Motion and the Classical Groups”. In: *Probability, Statistics and Their Applications: Papers in Honor of Rabi Bhattacharya*. Vol. 41. Institute of Mathematical Statistics, 2003, pp. 97–116.
- [7] P. Dey. “Limiting distribution of linear combination of coefficients of a representation w.r.t a random base”. 2008.
- [8] P. Diaconis. “Patterns in eigenvalues: the 70th Josiah Willard Gibbs lecture”. In: *Bull. Amer. Math. Soc.* 40.2 (2003), pp. 155–178.
- [9] P. Diaconis and S. Evans. “Linear functionals of eigenvalues of random matrices”. In: *Trans. Amer. Math. Soc.* 353 (2001), pp. 2615–2633.

- [10] P. Diaconis and D. Freedman. “A Dozen de Finetti-style Results in Search of a Theory”. In: *Annales de l’I. H. P., section B* 23 (1987), pp. 397–423.
- [11] P. Diaconis and M. Shahshahani. “On the eigenvalues of random matrices”. In: *J. Appl. Probab.* 31A (1994), pp. 49–62.
- [12] R. Durrett. *Probability: Theory and Examples*. 4th. Duxbury Press, 2005.
- [13] S. Evans. “Spectra of random linear combinations of matrices defined via representations and Coxeter generators of the symmetric group”. In: *Ann. Probab.* 37.2 (2009), pp. 726–741.
- [14] P. Hall and C. Heyde. *Martingale Limit Theory and Its Application*. New York, NY: Academic Press, 1980.
- [15] W. Hoeffding. “A combinatorial central limit theorem”. In: *Ann. Math. Statistics*. 22 (1951), pp. 558–566.
- [16] T. Jiang. “How many entries of a typical orthogonal matrix can be approximated by independent normals?” In: *Ann. Probab.* 34.4 (2006), pp. 1497–1529.
- [17] K. Johansson. “On random matrices from the compact classical groups.” In: *Annals of Mathematics*. 145 (1997), pp. 519–545.
- [18] D. L. McLeish. “Dependent Central Limit Theorems and Invariance Principles”. In: *Ann. Probab.* 2 (1974), pp. 620–628.
- [19] E. Meckes. “Linear functions on the classical matrix groups”. In: *Transactions of the American Mathematical Society* 360.10 (2008), pp. 5355–5366.
- [20] D. Petz and F. Hiai. *The Semicircle Law, Free Random Variables and Entropy*. Vol. 77. Mathematical Surveys and Monographs. Providence, RI: American Mathematical Society, 2000.

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